Optimality of linear optical Bell measurements

How much can ancillæ help?

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arXiv:1806.01243, Phys. Rev. A **98**, 042323

The task: unambiguous Bell measurement

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big) \qquad |\psi^+\rangle = \frac{1}{\sqrt{2}}\big(\,|01\rangle + |10\rangle\,\big)$$
$$|\phi^-\rangle = \frac{1}{\sqrt{2}}\big(\,|00\rangle - |11\rangle\,\big) \qquad |\psi^-\rangle = \frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\,\big)$$

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Projective measurement on the Bell basis

Unambiguous

Outcome never wrong, but can fail with probability $\mathcal{P}_{\text{fail}}$

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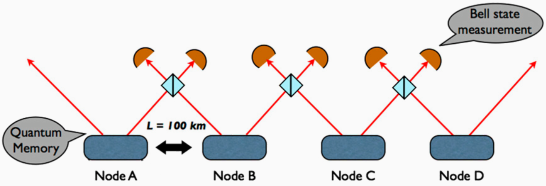
Quantum teleportation, dense coding, entanglement swapping...

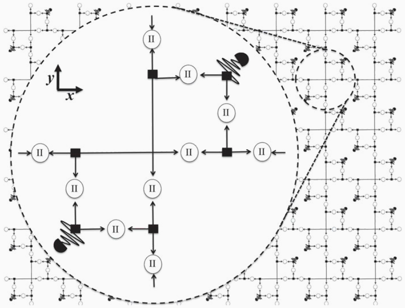
The applications



F. Sciarrino and P. Mataloni, PNAS Dec 2012, 109 (50) 20169-20170

Quantum networks

Quantum computation

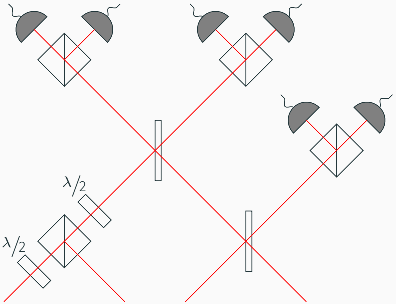


T. Rudolph, APL Photonics 2, 030901 (2017)

The framework: static linear optics

Why? Because...

- ...experimentally easier: no feedforward



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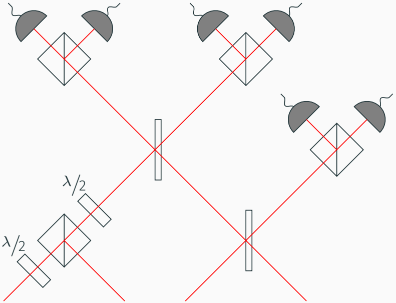
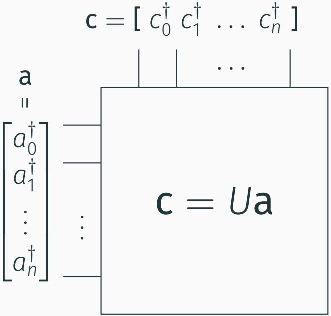
- ...simple mathematical framework to work with

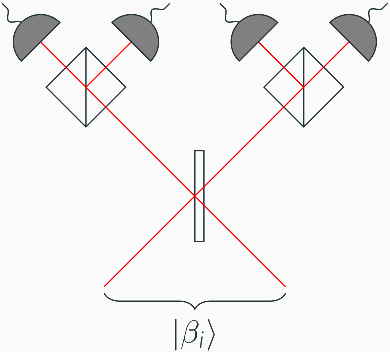




State of the art: Bell measurement without ancilla

[Braunstein and Mann, 1995]

- Known $\mathcal{P}_{\text{succ}} = \frac{1}{2}$ scheme [Braunstein & Mann, 1995]
- Optimal, even with feedforward [Calsamiglia & Lütkenhaus, 2001]



$$|\psi^+\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle)$$

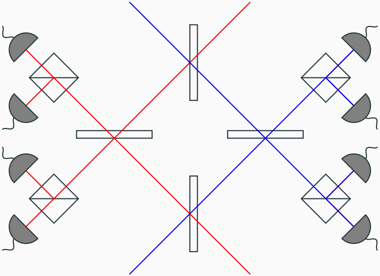
$$|\psi^-\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1001\rangle - |0110\rangle)$$

$$|\phi^\pm\rangle \longrightarrow \frac{i}{2}(\,|2000\rangle \pm |0200\rangle + |0020\rangle \pm |0002\rangle)$$

Beyond 1/2

[Grice, 2011]

- $\mathcal{P}_{\text{succ}} = \frac{3}{4}$ with an extra $|\phi^+\rangle$ as ancilla
- $\mathcal{P}_{\text{fail}} = 2^{-N}$ with GHZ-like states of $2^N - 2$ photons



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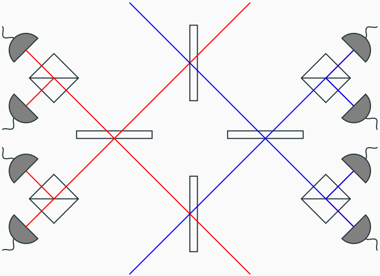

[Ewert and van Loock, 2014]

- $\mathcal{P}_{\text{succ}} = \frac{3}{4}$ with 4 single photons
- $\mathcal{P}_{\text{succ}} = \frac{25}{32}$ with 12 single photons
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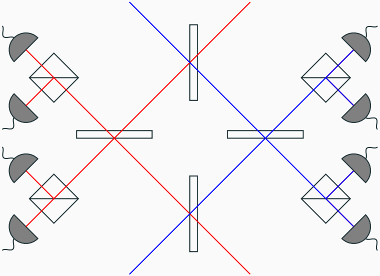

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What can we say about their optimality?

Our work

Ideally, we want an upper bound on $\mathcal{P}_{\text{succ}}$ depending on the ancilla.

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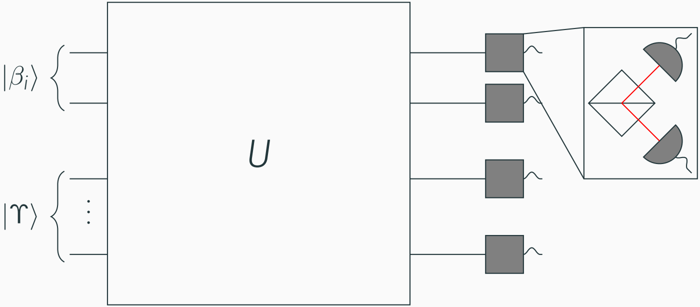
- An analytical bound for polarization-preserving interferometers
- A thorough numerical search for generic (small) interferometers

Analytical upper bound

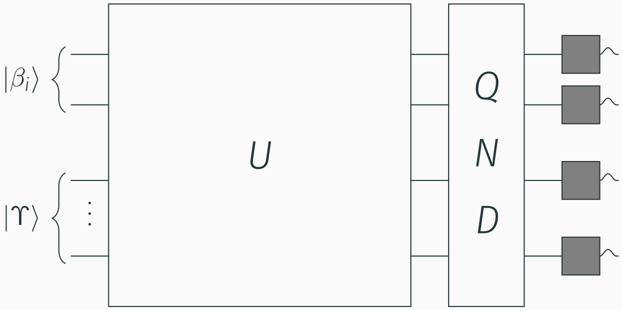
Overview of the proof



Input: $|\Psi_{\text{in}}\rangle = |\beta_i\rangle\,|\Upsilon\rangle$, with $|\Upsilon\rangle = \sum_{\lambda=0}^{k} v_\lambda\,|\Upsilon,\lambda\rangle$ a k-photon ancilla

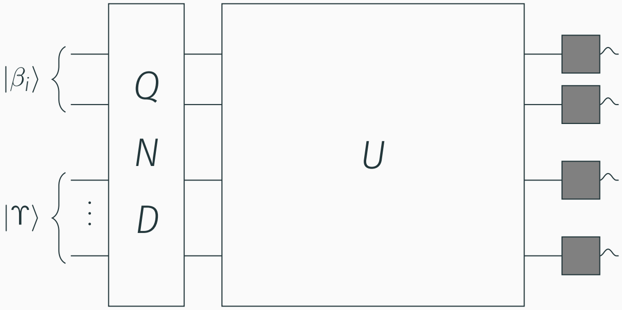
Each $|\Upsilon,\lambda\rangle$ is a state with λ horizontally polarized photons

Overview of the proof



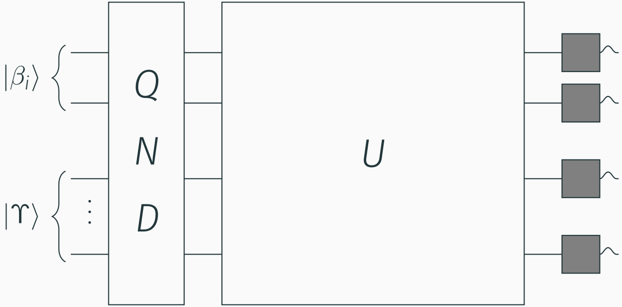
Output statistics unchanged under a projective measurement of the horizontally-polarized photon number of $|\Psi_{\text{out}}\rangle$

Overview of the proof



Measurement commutes $\iff$ polarization-preserving network

Overview of the proof



$|\psi^+\rangle$, $|\psi^-\rangle$ and $|\phi^\pm\rangle$ are mapped in three orthogonal subspaces $\forall\lambda$

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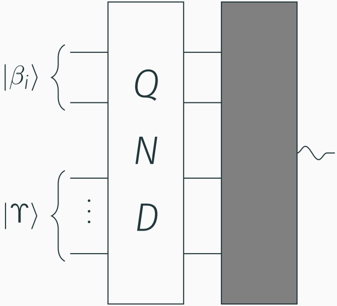

Non-orthogonal post-measurement states $|\Lambda^{\pm}\rangle$ corresponding to $|\phi^{+}\rangle$ and $|\phi^{-}\rangle$ can be discriminated with optimal $\mathcal{P}_{\text{succ}} \leq 1 - |\langle\Lambda^{+}|\Lambda^{-}\rangle|$.

Upper bounds

Ancilla polarization-based upper bound:

$$\mathcal{P}_{\text{fail}} \geq \frac{1}{2}\left(\max_{\lambda \text{ even}} |v_\lambda|^2 + \max_{\lambda \text{ odd}} |v_\lambda|^2 \right) \tag{1}$$

(looser) Photon number-based upper bound:

$$\mathcal{P}_{\text{fail}} \geq \frac{1}{\lceil k+1 \rceil_{\text{even}}} \tag{2}$$

N.B. The Grice's schemes saturate both bounds!

Linear network optimizer

Polynomial representation

Second quantization, n-modes interferometer.

Input/output state $\longrightarrow$ polynomial in the input/output mode operators:

$$|\psi_{\text{in}}\rangle = P_{\text{in}}(a_1^\dagger, \ldots, a_n^\dagger)\,|0\rangle \qquad |\psi_{\text{out}}\rangle = P_{\text{out}}(c_1^\dagger, \ldots, c_n^\dagger)\,|0\rangle$$

Action of the interferometer on $|\psi_{\text{in}}\rangle \longrightarrow$ unitary transformation U acting on the mode operators:

$$a_i^\dagger = \sum_{j=1}^{n} u_{ij}\, c_j^\dagger$$

Measurement of $|\psi_{\text{out}}\rangle$ by an array of PNRD.
Detection event $\longrightarrow$ a configuration of clicks at the output.

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4. A f.o.m. $f(U)$ is numerically optimized over $U(n)$

An example: no ancilla on $|\phi^+\rangle$

$$n = 4,\, k = 0 \qquad P_{\text{in}} = \frac{1}{\sqrt{2}}(a_1^\dagger a_3^\dagger + a_2^\dagger a_4^\dagger)$$

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$$\downarrow$$

$$P_{\text{out}} = \frac{1}{\sqrt{2}}\Big(\sum_{j_1} u_{1j_1} c_{j_1}^\dagger\Big)\Big(\sum_{j_2} u_{3j_2} c_{j_2}^\dagger\Big) + \frac{1}{\sqrt{2}}\Big(\sum_{j_3} u_{2j_3} c_{j_3}^\dagger\Big)\Big(\sum_{j_4} u_{4j_4} c_{j_4}^\dagger\Big)$$

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2000	$u_{11}u_{31} + u_{21}u_{41}$
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1100	$(u_{11}u_{32} + u_{12}u_{31} + u_{21}u_{42} + u_{22}u_{41})/\sqrt{2}$
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Optimization: a case for symbolic computation

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For (small!) $\frac{3}{4}$ Grice's scheme:

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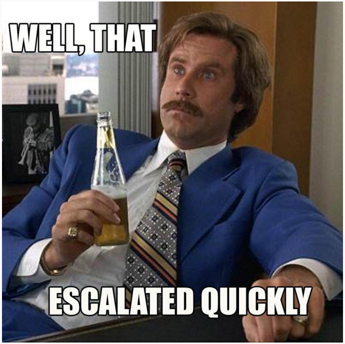
But 2^{nd} iteration, $\frac{7}{8}$:

Total number of functions	490314
Independent	22
Total number of terms	$1{,}8 \cdot 10^6$



Results

Ancilla: a single photon

Simplest type of ancilla, no known scheme

- Polarization-preserving bound predicts $\mathcal{P}_{\text{succ}} \leq \frac{1}{2}$
- Numerical search confirms the result

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We find no advantage using just one extra photon.

More single, unentangled photons

k photons	Pol. Pres. bound	Num. search	Explicit scheme
2	5/8	5/8	5/8
4	3/4	3/4	3/4
6	13/16	3/4	—
8	25/32	—	49/64
12	13/16	—	25/32

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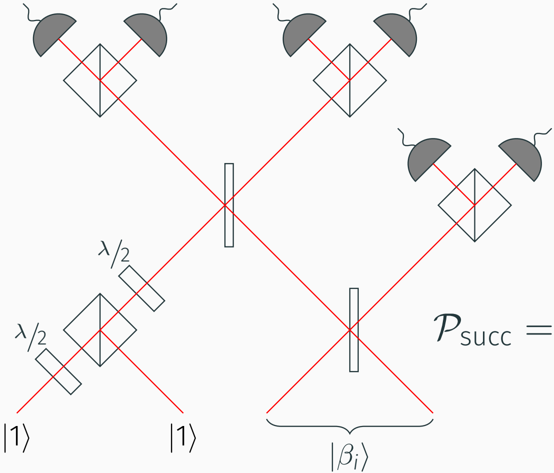
just 2 single photons beat 50% limit!

The “half”-Ewert & van Loock scheme



$$\mathcal{P}_{\text{succ}} = \frac{5}{8}$$

Also independently found by Ewert & van Loock

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  - Pol. pres. bound predicts $\mathcal{P}_{\text{fail}} \gtrsim \dfrac{1}{\sqrt{\pi k}} > \dfrac{1}{k}$
- Grice's $\frac{7}{8}$ scheme needs $n = 16, k = 6$: barely out of reach

Conclusion

Achievements

What can we say on the optimal $\mathcal{P}_{\text{succ}}$ of unambiguous Bell measurement?

- Upper bound for polarization-preserving interferometers, saturated by known schemes
- Hybrid numerical/symbolical search
  - Confirms optimality of (some) known schemes
  - New 2-photon scheme with $\mathcal{P}_{\text{succ}} = \frac{5}{8}$
- Explored several ancillæ$\longrightarrow\frac{3}{4}$ stays the best (for small networks)

Open questions

- Include noise. But how to generalize “unambiguous”?
- Adapt for state generation
- Relax PNRD assumption?

Thank you